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THE ONSET OF INSTABILITY IN A HORIZONTAL FLUID LAYER DUE TO A STEP CHANGE IN TEMPERATURE

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Abstract—The onset of instability in a fluid layer which is subjected to a sudden change in surface temperature is analysed by a modified version of the frozen time hypothesis. The assumption that for large Prandtl number the temperature disturbances are confined to the effective thermal depth leads to a considerable simplification in the formulation of the stability problem. The effect of the Rayleigh number on the onset time is discussed and clearly explained. The relation between the Rayleigh number and the wavenumber predicted here agrees remarkably well with the extant amplification theory.

NOMENCLATURE

a,	horizontal dimensionless wavenumber;
a_x, a_y,	components of a;
$b_k^{(n)}$,	coefficients of the power series;
C_n,	constants used in the general solution of the perturbation equations;
D,	$d/d\zeta$;
f_n,	rapidly convergent power series;
g,	gravitational acceleration;
h,	depth of the fluid layer;
p,	dimensionless pressure;
Pr,	Prandtl number;
Ra,	Rayleigh number;
t,	dimensionless time;
t_f,	dimensionless frozen time;
T_0,	initial temperature;
T_1,	$T_0 - \Delta T$;
$\mathbf{u}$,	dimensionless velocity vector;
u, v, w,	dimensionless velocity components;
$w_i(\zeta)$,	$w^*(z)$ for $0 \leqslant \zeta \leqslant 1$;
$w_0(\zeta)$,	$w^*(z)$ for $1 \leqslant \zeta \leqslant 1/\delta$;
x, y, z,	dimensionless rectangular coordinates.

Greek symbols

β,	δa;
ΔT,	temperature difference;
δ,	dimensionless effective thermal depth;
ζ,	z/δ;
θ,	dimensionless temperature difference;
θ'_{bu},	dimensionless temperature gradient at the upper surface;
κ,	thermal diffusivity;
μ,	viscosity;
ν,	kinematic viscosity;
ρ,	density;
ρ_0,	density at the initial temperature;
σ,	growth rate of the disturbance.

Superscripts

$'$,	dimensional quantity, derivative with respect to ζ;
$*$,	amplitude of the disturbance at a frozen time.

Subscripts

b,	basic state;
c,	critical condition;
1,	perturbation quantity.

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1. INTRODUCTION

WHEN a fluid layer which is originally isothermal and quiescent is suddenly cooled from above (or heated from below) with the corresponding Rayleigh number exceeding the critical value, the top-heavy fluid layer becomes unstable and motion begins. The determination of the time at which this convective motion starts constitutes an important stability problem, where the time-dependent nonlinear base temperature profile is concerned.

Lick [1] and Currie [2] analysed this problem by adopting a quasi-steady model in which the nonlinear base temperature profile was considered to be frozen at each instant in time and was approximated by two linear segments. Foster [3], Mahler *et al.* [4] and Gresho and Sani [5] questioned this frozen time analysis and applied initial value techniques under the assumption that the critical state is attained at the time when the fastest growing initial disturbance has been amplified by several orders of magnitude.

In their investigations on the onset of cellular convection in a flowing liquid layer, Choi [6] and Davis and Choi [7] applied both the local stability theory and the amplification theory. In the latter the spatial growth of the disturbances is considered and in the former the spatially developing base temperature profile is treated locally as being frozen at each axial position. The best agreement with experiment has been

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obtained by means of the modified local stability analysis with an additional assumption that for large Prandtl number the cellular convection is confined to the thermal boundary layer at the onset of instability.

Since the above problem is similar to the present one except that the base temperature is spatially developing instead of time dependent, their modified concept can be extended to our problem without loss of generality. In fact, it is the purpose of the present study to re-examine the assumptions of the quasi-steady base temperature profile and the confinement of the temperature disturbance to the effective thermal depth.

2. STABILITY ANALYSIS

Consider a fluid layer which is originally isothermal and quiescent. It is considered that the fluid is infinite in horizontal directions but bounded at top and bottom by parallel rigid conducting plates. At some time $t' = 0$, the upper surface is suddenly cooled to a temperature ΔT below the original value and from then on maintained there, while the lower surface is always held at fixed temperature. If the corresponding Rayleigh number exceeds the critical value, then at a later time the fluid will eventually become unstable and motion will begin. This onset time can be found by solving a stability problem, where a time-dependent nonlinear base temperature profile is assumed as shown schematically in Fig. 1. The coordinate z' will be measured vertically downwards from the top surface and the coordinates x' and y' lie in the horizontal planes.

The starting point for the present problem is the derivation of the governing equations for a mechanically incompressible Newtonian fluid with negligible dissipation energy on the basis of Boussinesq approximation [8]. After introducing the following dimensionless variables:

$$(x, y, z) = \frac{1}{h}(x', y', z'),$$

$$(\mathbf{u}; u, v, w) = \frac{h}{\kappa}(\mathbf{u}'; u', v', w'),$$

$$p = \frac{(p' - \rho_0 g z')h^2}{\mu\kappa}, \quad \theta = \frac{T' - T_1}{\Delta T}, \quad t = \frac{t'\kappa}{h^2},$$

the governing equations can be made dimensionless as follows:

$$\nabla \cdot \mathbf{u} = 0, \tag{1}$$

$$\frac{1}{Pr}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P + \nabla^2\mathbf{u} - Ra\theta\mathbf{k} + Ra\mathbf{k}, \tag{2}$$

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \nabla^2\theta. \tag{3}$$

When the sudden change in temperature is applied it is expected that at first there will be no motion due to natural convection and heat will be transferred by conduction only. Therefore, in the basic state $\mathbf{u}_b \equiv \mathbf{0}$ and $\theta_b(z, t)$ satisfies the following transient heat conduction equation:

$$\frac{\partial\theta_b}{\partial t} = \frac{\partial^2\theta_b}{\partial z^2}, \tag{4}$$

with the initial and the boundary conditions

$$\begin{aligned} \theta_b &= 1 \quad \text{for } 0 \leqslant z \leqslant 1,\ t = 0, \\ \theta_b &= 0 \quad \text{for } z = 0,\ t > 0, \\ \theta_b &= 1 \quad \text{for } z = 1,\ t \geqslant 0. \end{aligned} \tag{5}$$

The solution to equation (4) subject to conditions (5) is easily obtained by the method of separation of variables (Graetz type) and is

$$\theta_b = z + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi z}{n}\exp(-n^2\pi^2 t). \tag{6}$$

However, at small times (say $t < \sim 0.01$) this solution converges very slowly and thus the following approximation (Leveque type) based on the fluid having infinite depth is known to be more useful [5]:

$$\theta_b = \operatorname{erf}[z/(4t)^{1/2}]. \tag{7}$$

The perturbation quantities are superimposed on the basic quantities in the form

$$(\mathbf{u}, p, \theta) = (\mathbf{u}_1, p_b + p_1, \theta_b + \theta_1). \tag{8}$$

Substituting equation (8) into equations (1), (2) and (3) and applying the linear stability theory we obtain the perturbation equations. To eliminate the pressure term from the momentum equation, it is convenient to take a double curl of that equation with the use of continuity equation. The z-component of the resulting momentum equation and the energy equation for the perturbation quantities are given as follows [5]:

$$\left(\frac{1}{Pr}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w_1 = -Ra\nabla_2^2\theta_1, \tag{9}$$

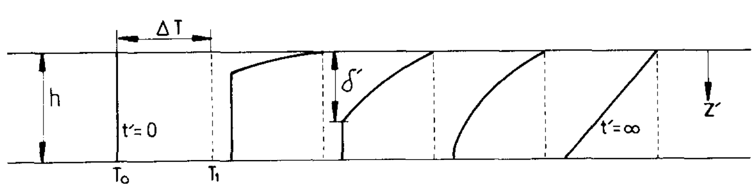

FIG. 1. Schematic diagram of the system under consideration.

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta_1 = -w_1\frac{\partial\theta_b}{\partial z}, \quad (10)$$

with the boundary conditions

$$w_1 = \frac{\partial w_1}{\partial z} = \theta_1 = 0 \quad \text{at } z = 0, 1 \quad (11)$$

where

$$\nabla_2^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the 2-dim. operator.

Noting that there are no lateral boundaries, we can express an arbitrary disturbance in the x–y plane in terms of 2-dim. periodic waves

$$(w_1, \theta_1) = [\bar{w}_1(z, t), \bar{\theta}_1(z, t)]\exp[i(a_x x + a_y y)] \quad (12)$$

where $a = (a_x^2 + a_y^2)^{1/2}$ is the horizontal wavenumber of the disturbance.

3. MODIFIED FROZEN TIME ANALYSIS

Since $\partial\theta_b/\partial z$ is a function of t as well as z, the variables z and t in equations (9) and (10) may not be separable. The early researchers [1, 2] of the present problem resorted to the so-called frozen time analysis, in which the non-linear base temperature profile θ_b is frozen at each instant in time so that $\partial\theta_b/\partial z$ becomes a function of z only and t is considered as a parameter. It is then possible to separate the variables z and t in the differential equations (9) and (10) so that the disturbances can be expressed as

$$(w_1, \theta_1) = [w_1^*(z), \theta_1^*(z)]\exp(\sigma t) \times \exp[i(a_x x + a_y y)]. \quad (13)$$

It was assumed by Currie [2] that the principle of exchange of stability is valid at the onset of instability. Thus based on the marginal state method of the frozen time analysis the equations were written in the form

$$\left(\frac{d^2}{dz^2} - a^2\right)^2 w_1^* + Ra\,a^2\theta_1^* = 0 \quad (14)$$

$$\left(\frac{d^2}{dz^2} - a^2\right)\theta_1^* - \frac{\partial\theta_b}{\partial z}w_1^* = 0 \quad (15)$$

with the boundary conditions

$$w_1^* = \frac{dw_1^*}{dz} = \theta_1^* = 0 \quad \text{at } z = 0, 1. \quad (16)$$

Foster [3], Mahler *et al.* [4] and Elder [9] took entirely different approaches dependent on initial value techniques. Particularly, Gresho and Sani [5] compared the initial value techniques and the frozen time analyses by using Galerkin's method to obtain approximate solutions. They argued that the marginal state assumption based on the frozen time concept is not valid since the decay rate of the base temperature transient is large at the onset of instability. Assuming some initial disturbance typically 'white noise', they considered that the critical state is attained when it has been amplified by several orders of magnitude. But there seems to be some room for improvement in their argument since their results turned out to be substantially affected by the choice of the initial disturbances which are not uniquely defined. However, when we consider their results from another point of view, it is quite noteworthy that their results for $Pr = 7$ indicated that the temperature disturbances near the onset time ($t = 0.02$ for $Ra = 10^5$) are independent of the lower boundary and distributed in the effective thermal depth whereas the velocity disturbances are controlled by the boundary conditions at both surfaces.

Studies which are related to the present problem and of much importance are those of Choi [6] and Davis and Choi [7], who took notice of this fact in their theoretical and experimental investigations on the onset of cellular convection in a flowing liquid layer. They applied both local stability analysis and amplification theory, as explained in the latter part of Section 1. The best agreement with experiment was obtained by means of the modified local stability analysis which assumes that for large Prandtl number, say $Pr > 6$, the temperature disturbances are confined to the thermal boundary layer at the onset of instability.

In the present study, we shall restrict ourselves to the limiting case of large Prandtl number ($Pr \to \infty$) based on the results of Choi [6] and Davis and Choi [7]. Then under this condition, it can be assumed that at the onset of thermal instability the temperature disturbances are confined to the effective thermal depth before the effect of the base temperature penetrates the whole fluid layer. We readily recall that this is also consistent with what Gresho and Sani [5] observed for $Pr = 7$ as previously discussed. Consequently, it can be understood that the classical frozen time concept becomes meaningful again in this case.

On the other hand, for the opposite case of $Pr \to 0$, the conduction effect accompanied by convection may exist outside the effective thermal depth at the moment when the motion sets in. In this case the propagation of temperature disturbances θ_1^* may affect the full depth. In fact, it is well known that for mercury ($Pr \simeq 0.025$) no stationary regular motion in the form of hexagonal cells or rolls can be observed at the onset of thermal instability. The applicability of the frozen time model is then severely restricted and should be re-examined for this case as Gresho and Sani suspected.

Thus, based on the modified frozen time concept for large Pr, it naturally follows that the principle of exchange of stability is valid at the onset of instability, i.e., that equations (14)–(16) still hold. And the modified frozen time concept for large Pr also leads us to make the following very useful assumption:

$$\theta_1^* = 0 \quad \text{for } z \geqslant \delta, \quad (17)$$

where δ is the effective thermal depth. In the present study δ is taken as the depth from the top surface to where $\theta_b = 0.99$. Paying attention to the equation (17) and eliminating θ_1^* from the system of equations (14) and (15) for $z \leqslant \delta$, we can reformulate the perturbation equations as follows:

$$\left(\frac{d^2}{dz^2} - a^2\right)^3 w_1^* + a^2 Ra \frac{\partial \theta_b}{\partial z} w_1^* = 0 \quad \text{for } z \leqslant \delta, \tag{18}$$

$$\left(\frac{d^2}{dz^2} - a^2\right)^2 w_1^* = 0 \quad \text{for } z \geqslant \delta \tag{19}$$

with the boundary conditions

$$w_1^* = \frac{dw_1^*}{dz} = \left(\frac{d^2}{dz^2} - a^2\right)^2 w_1^* = 0 \quad \text{at } z = 0, \tag{20}$$

$$w_1^* = \frac{dw_1^*}{dz} = 0 \quad \text{at } z = 1, \tag{21}$$

and the interface conditions that

$$w_1^*, \frac{dw_1^*}{dz}, \frac{d^2 w_1^*}{dz^2}, \frac{d^3 w_1^*}{dz^3}$$

$$\text{and } \frac{d^4 w_1^*}{dz^4} \text{ are continuous at } z = \delta. \tag{22}$$

Here, the conditions (22) are established on the continuity of the solution including velocity, stress and temperature at the edge of the effective thermal depth. In the next section, we shall show a very convenient way of approximating θ_b through which the solutions can be represented as sums of fast converging series and the critical values can be readily calculated.

4. METHOD OF SOLUTION

The fact that $\partial\theta_b/\partial z$ is a function of z still complicates the solution to the system of equations (18)–(22). In order to overcome this difficulty, Lick [1] and Currie [2] employed a useful approximation with which they represented the basic temperature profile by two straight line segments. In the present study however we consider that an even better approximation which is simple to use and yet preserves all the necessary features of the exact solutions can be obtained in the power form as follows:

$$\theta_b = 1 - \left(1 - \frac{z}{\delta}\right)^N \quad \text{for } 0 \leqslant z \leqslant \delta, \tag{23}$$

$$\theta_b = 1 \quad \text{for } \delta \leqslant z \leqslant 1, \tag{24}$$

where N is an appropriate exponent. After differentiating the equation (23) with respect to z and considering the boundary conditions, this exponent can be determined as

$$N = \theta'_{bu}\,\delta, \tag{25}$$

where θ'_{bu} is the temperature gradient at the upper surface. The numerical values of δ and θ'_{bu} for a frozen time t_f are calculated from the exact solution of the basic temperature profile, i.e. equation (6) or (7) depending on the range of t_f

$$\delta = 3.64\, t_f^{1/2}, \quad \theta'_{bu} = \frac{1}{(\pi t_f)^{1/2}} \quad \text{for } t_f < 0.01, \tag{26}$$

$$\delta = z|_{\theta_b = 0.99}, \quad \theta'_{bu} = 1 + 2\sum_{n=1}^{\infty} \exp(-n^2\pi^2 t_f) \quad \text{for } t_f \geqslant 0.01. \tag{27}$$

In Fig. 2 a comparison is made between the exact solution and the present modified solution. We note that there is only negligible difference between them and that the present modification is more practical than the two segment approximation of Currie [2]. It is now convenient to rewrite equations (18)–(22) by introducing a new dimensionless variable $\zeta = z/\delta$. In doing so, we use the Taylor series expansion of the term $\partial\theta_b/\partial z$ with respect to ζ and make the following definitions:

$$w_i(\zeta) = w_1^*(z) \quad \text{for } 0 \leqslant \zeta \leqslant 1, \qquad w_0(\zeta) = w_1^*(z) \quad \text{for } 1 \leqslant \zeta \leqslant 1/\delta. \tag{28}$$

The perturbation equations now become

$$(D^2 - a^2\delta^2)^3 w_i + Ra\, a^2 \delta^6 \theta'_{bu} \left\{1 + \sum_{n=1}^{\infty} (-1)^n \times \left[\prod_{j=1}^{n} \left(\frac{\theta'_{bu}\delta}{j} - 1\right)\right] \zeta^n\right\} w_i = 0 \quad \text{for } 0 \leqslant \zeta \leqslant 1, \tag{29}$$

$$(D^2 - a^2\delta^2)^2 w_0 = 0 \quad \text{for } 1 \leqslant \zeta \leqslant 1/\delta \tag{30}$$

with the boundary and interface conditions

$$w_i = Dw_i = (D^2 - a^2\delta^2)^2 w_i = 0 \quad \text{at } \zeta = 0, \tag{31}$$

$$w_0 = Dw_0 = 0 \quad \text{at } \xi = 1/\delta, \tag{32}$$

$$w_i - w_0 = D^n w_i - D^n w_0 = 0 \,(n = 1, 2, 3, 4) \quad \text{at } \xi = 1, \tag{33}$$

where D denotes $d/d\zeta$.

A general solution of the above problem can be constructed in the form

$$w_i = \sum_{n=0}^{5} C_n f_n(\zeta), \qquad w_0 = (C_6 + C_7\zeta)\, e^{-\delta a \zeta} + (C_8 + C_9\zeta)\, e^{\delta a \zeta} \tag{34}$$

where C_n $(n = 0, 1, 2, \ldots, 9)$ are arbitrary constants and $f_n(\zeta)$ are rapidly convergent power series [10],

$$f_n(\zeta) = \sum_{k=0}^{\infty} b_k^{(n)} \zeta^k \quad (n = 0, 1, 2, \ldots, 5). \tag{35}$$

The series coefficients for $k \leqslant 5$ are specified as

$$b_{-1}^{(n)} = 0, \qquad b_k^{(n)} = \delta_{kn} \quad \text{for } k = 0, 1, 2, \ldots, 5, \tag{36}$$

and those for $k \geqslant 6$ are determined in terms of the preceding coefficients obeying the recurrence formulas generated from equation (18)

$$b_k^{(n)} = \frac{(\delta a)^2}{k(k-1)} \left\{3 b_{k-2}^{(n)} + \frac{(\delta a)^2}{(k-2)(k-3)}\right.$$

$$\times\left[-3b_{k-4}^{(n)}+\frac{(\delta a)^2}{(k-4)(k-5)}b_{k-6}^{(n)}\right]\Bigg\}$$

$$-\frac{(k-6)!}{k!}Ra\,a^2\delta^6\theta'_{bu}\left\{b_{k-6}^{(n)}+\sum_{m=-1}^{k-7}(-1)^{k-6-m}\right.$$

$$\left.\times\left[\prod_{j=1}^{k-6-m}\left(\frac{\delta\theta'_{bu}}{j}-1\right)\right]b_m^{(n)}\right\}. \quad (37)$$

The constants $C_n (n = 0, 1, 2, \ldots, 9)$ are chosen to satisfy the boundary and the interface conditions. From the boundary conditions at $\zeta = 0$, we obtain

$$C_0 = C_1 = 0,\quad C_4 = \frac{(\delta a)^2}{6}C_2. \quad (38)$$

From the conditions at $\zeta = 1/\delta$ and $\zeta = 1$, we obtain 7 algebraic equations in terms of 7 constants, $C_2, C_3, C_5, C_6, C_7, C_8$ and C_9. Thus in order that there exist nontrivial solutions for these constants the following secular equation must be satisfied:

$$\begin{vmatrix} 0 & 0 & 0 & e^{-a} & \frac{1}{\delta}e^{-a} & e^{a} & \frac{1}{\delta}e^{a} \\ 0 & 0 & 0 & -\beta e^{-a} & (1-a)e^{-a} & \beta e^{a} & (1+a)e^{a} \\ -\left[f_2(1)+\frac{\beta^2}{6}f_4(1)\right] & -f_3(1) & -f_5(1) & e^{-\beta} & e^{-\beta} & e^{\beta} & e^{\beta} \\ -\left[f'_2(1)+\frac{\beta^2}{6}f'_4(1)\right] & -f'_3(1) & -f'_5(1) & -\beta e^{-\beta} & (1-\beta)e^{-\beta} & \beta e^{\beta} & (1+\beta)e^{\beta} \\ -\left[f''_2(1)+\frac{\beta^2}{6}f''_4(1)\right] & -f''_3(1) & -f''_5(1) & \beta^2e^{-\beta} & \beta(\beta-2)e^{-\beta} & \beta^2e^{\beta} & \beta(\beta+2)e^{\beta} \\ -\left[f'''_2(1)+\frac{\beta^2}{6}f'''_4(1)\right] & -f'''_3(1) & -f'''_5(1) & -\beta^3e^{-\beta} & \beta^2(3-\beta)e^{-\beta} & \beta^3e^{\beta} & \beta^2(3+\beta)e^{\beta} \\ -\left[f^{iv}_2(1)+\frac{\beta^2}{6}f^{iv}_4(1)\right] & -f^{iv}_3(1) & -f^{iv}_5(1) & \beta^4e^{-\beta} & \beta^3(\beta-4)e^{-\beta} & \beta^4e^{\beta} & \beta^3(\beta+4)e^{\beta} \end{vmatrix} = 0. \quad (39)$$

Here $\beta = \delta a$.

A neutral stability curve can be generated by solving the above problem for any frozen time t_f (hence for given values of δ and θ'_{bu}). Figure 3 shows a composite of such neutral stability curves for various frozen times. The minimum Rayleigh number on each curve is considered to be the critical Rayleigh number above which thermal instability occurs at the corresponding frozen time. Or equivalently, it can be said that the time required for the onset of thermal instability under given Rayleigh number may be predicted by finding the stability curve on which the Rayleigh number is the minimum.

For the onset time $t_c = 1.0$, the critical Rayleigh number $Ra_c = 1707.7$ and the critical wavenumber $a_c = 3.117$. This is an indication that as $t_c \to \infty$, the classical Rayleigh–Bénard problem which corresponds to the case of a uniform temperature gradient is recovered. The fact that Ra_c has a minimum of approximately 1670 near $t_c = 0.1$ may be compared with the result of Currie [2] who obtained $\min(Ra_c) = 1340$ at $\delta = 0.72$ (equivalent to $t_c = 0.04$). However, this is of only limited interest in the present study since we are more interested in the range of Rayleigh numbers much greater than 1708 (Fig. 4).

Once the onset conditions are determined, the distributions of the velocity and temperature disturbances can be calculated only up to multiplicative constants and we can proceed to compare our theory with those of others.

5. RESULTS AND DISCUSSION

The effect of Rayleigh number on onset time can be studied by the method discussed in the previous section. The result of the present analysis is shown in Fig. 4 together with the results of Gresho and Sani [5] and Currie [2].

For a given Rayleigh number the onset time predicted by the present study is shown to be smaller than that by the amplification theory of Gresho and Sani for $Pr = 7$ but larger than those by the frozen time analysis of theirs and the two segment approximation of Currie. In other words the asymptotic relations for Ra_c and t_c are: $Ra_ct_c^{1.5} = 280$ in the amplification theory, $Ra_ct_c^{1.5} = 14.2$ in the present study and $Ra_ct_c^{1.5} = 2$ in the frozen time analysis. Currie's representation of the base temperature profile by two straight line segments may be adequate for very small and for very large onset times. But no matter how well the two segments are chosen for intermediate onset times, there is always much discrepancy from the actual profiles, especially near the upper surface as can be conjectured from Fig. 2. Gresho and Sani used a time dependent Galerkin method in which the temperature and velocity disturbances are each represented by a series of specified trial functions with time dependent coefficients. Although the exact form of the base temperature profile is used throughout their analysis,

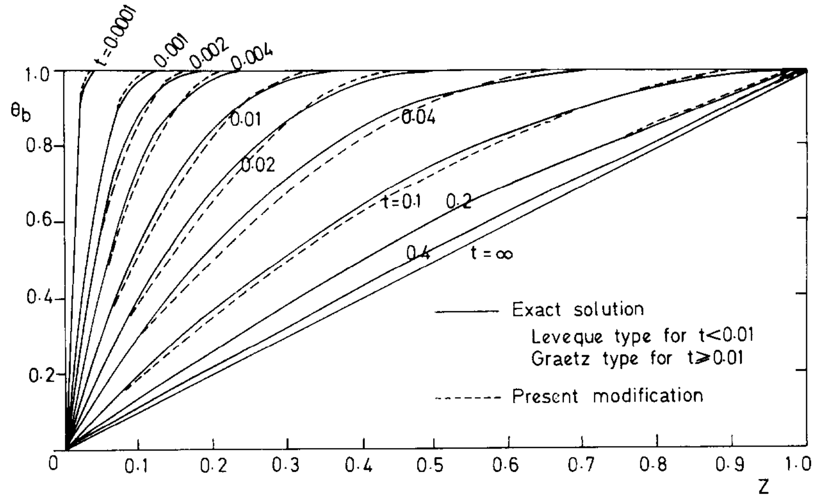

FIG. 2. Comparison of exact and modified base temperature profiles.

their results may still depend on the forms and the number of terms of the trial functions used. Above all, it should be noted that in the amplification theory they considered the critical state to be attained at the time the fastest growing disturbance has grown sufficiently to be observed, say 10^3 times the initial value. This definition lacks uniqueness since it involves the measurability of discernible motion. However, when we consider that in the present modified frozen time analysis, the onset time is defined as the time the fastest growing disturbance is neutrally stable, the result of their amplification theory is not altogether irrelevant to ours. That is, the discrepancy between the two analyses can be attributed to the time for the initial disturbance defined in the amplification theory to grow to a certain observable size.

In the numerical study on the temporal development of a model of high Rayleigh number convection, Elder [11] assumed that when $Pr \geq 1$ convective flows behave as if $Pr = \infty$ and found that the critical time (onset time) is 2.65×10^{-3} at $Ra = 10^5$ for a fluid layer suddenly heated from below. When this value is compared with ours in Fig. 4, it can be easily known that the agreement is very good. This is another and more direct manifestation of the validity of our assumption.

Let us examine one additional set of information on the wavenumber. In Fig. 5 the effect of Rayleigh number on wavenumber is shown. The conventional frozen time analysis predicts a flattening of the wavenumber *vs* Rayleigh number curve for a considerable range of Rayleigh numbers. This is not considered to be the proper dependence of a_c on Ra_c, since it is generally known that the size of the disturbance becomes smaller as Ra_c increases. However, the amplification theory and the modified frozen time analysis

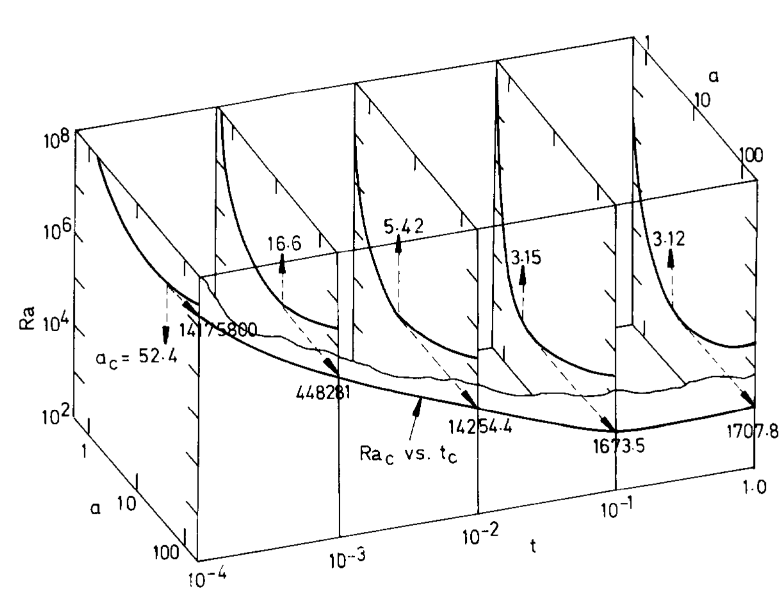

FIG. 3. Neutral stability curves for various times based on modified frozen time analysis.

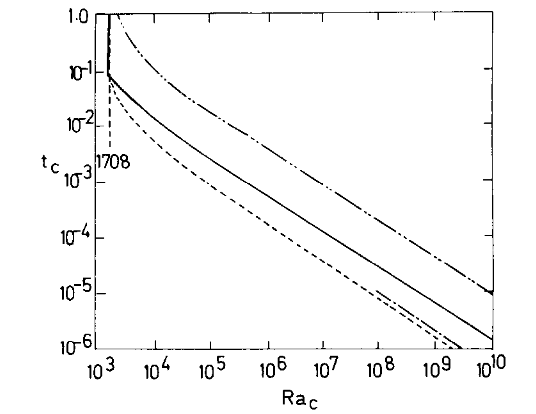

FIG. 4. Effect of Rayleigh number on onset time: —— modified frozen time analysis, ---- frozen time analysis of Gresho and Sani [5], —·–·— amplification theory of Gresho and Sani [5] for Pr = 7, —·— two segment approximation of Currie [2].

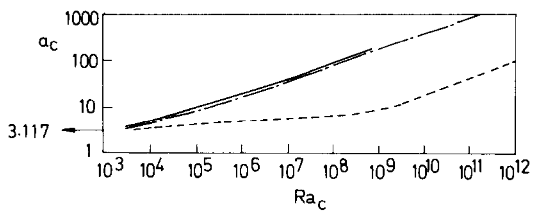

FIG. 5. Effect of Rayleigh number on wavenumber: —— modified frozen time analysis, ---- frozen time analysis of Gresho and Sani [5], —·— amplification theory of Gresho and Sani [5].

are in fair agreement with each other and predict the proper dependence of a_c on Ra_c, that is, the increase of a_c with Ra_c. The common asymptotic solution in these two analyses is $Ra_c = 250a_c^3$. The fact that the same a_c is obtained for a given Ra_c in spite of the different definitions of the onset time in the two analyses is a very remarkable and encouraging result which strongly supports the previous reasoning on the difference in the onset times.

Summarizing the discussions thus far, the assumption that for large Prandtl number the temperature disturbances are confined to the effective thermal depth is proven to be valid in the case where a sudden change in temperature is applied to a boundary of a fluid layer.

Once the critical values are determined for a given onset time, the distribution of disturbances can be calculated only up to multiplicative constants. For example, the solutions, θ_1^* and w_1^* for t_c = 0.001, 0.01 and 0.1 are obtained by simply taking C_2 = 1 and displayed in Figs. 6 and 7, respectively, where they are normalized with respect to the maximum values represented as scale factors. It is recalled that in Fig. 2 of Gresho and Sani [5], the temperature disturbance beyond the effective thermal depth is negligible. Then it is of particular interest to note that θ_1^* of the present solution for t_c = 0.01 has the same general nature as that of Gresho and Sani throughout the most part of the effective thermal depth except near the outer edge.

6. CONCLUSIONS

The onset of thermal instability in a fluid layer undergoing a step change in temperature has been investigated by applying the marginal state method of the modified frozen time analysis. The analysis has been carried out under the assumption that for large Prandtl number, say for $Pr > 6$, the temperature disturbances are confined to the effective thermal depth at the onset of thermal instability.

This assumption has led in the first place to a considerable simplification in the formulation of the stability problem, where the solutions can be found in the form of rapidly convergent power series. The neutral stability curve for any frozen time has been

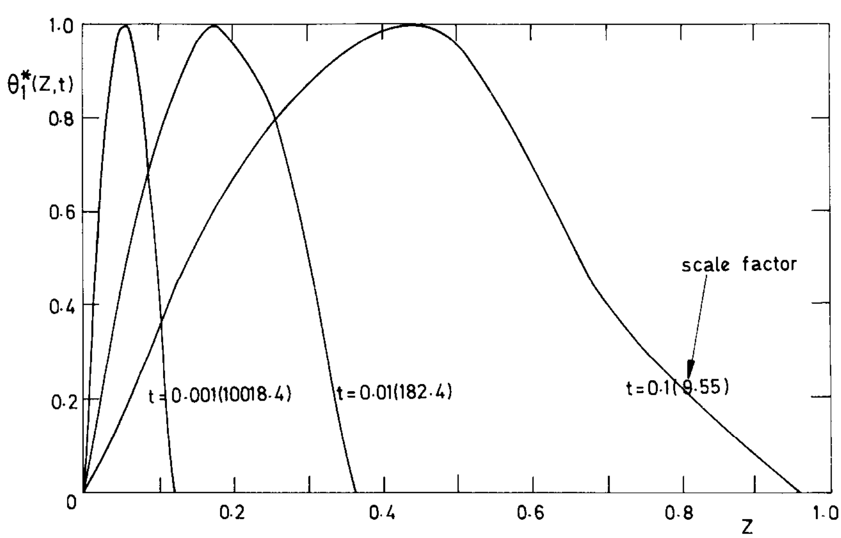

FIG. 6. Distribution of temperature disturbance at various times.

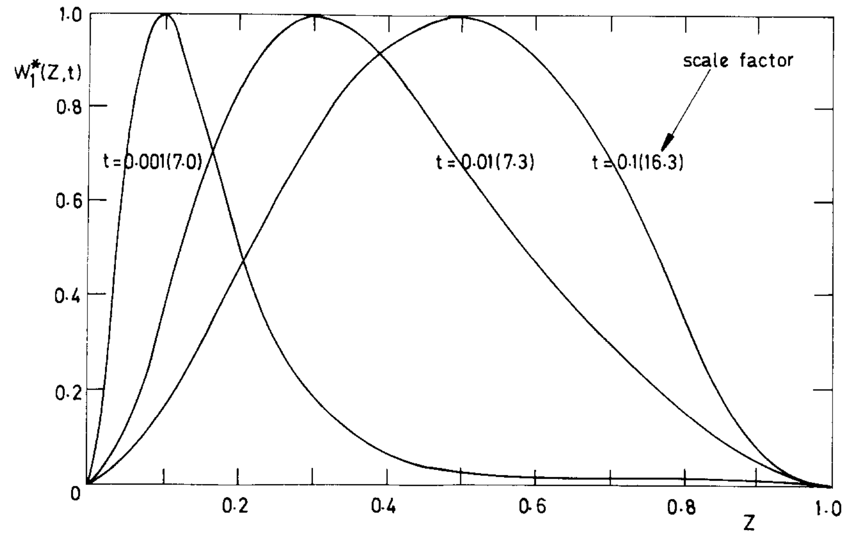

FIG. 7. Distribution of z-component velocity disturbance at various times.

then obtained by solving a secular equation generated from the boundary conditions.

The present theory predicts the onset time for a given Rayleigh number to be smaller than that predicted by the conventional amplification theory. A lucid interpretation of this discrepancy has been given by comparing the definitions of the onset times. The present theory also predicts that the wavenumber increases with the Rayleigh number, which agrees remarkably well with the conventional amplification theory. This agreement has substantiated that the present theory is valid and complements the extant theories.

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APPARITION DE L'INSTABILITE DANS UNE COUCHE FLUIDE HORIZONTALE SOUMISE A UN CHANGEMENT ECHELON DE TEMPERATURE

Résumé—L'apparition de l'instabilité d'une couche fluide soumise à un brusque changement de température superficielle est analysée par une version modifiée de l'hypothèse du temps gelé. L'hypothèse, selon laquelle pour un grand nombre de Prandtl les perturbations de température sont confinées à une profondeur thermique effective, conduit à une simplification considérable dans la formulation du problème de stabilité. L'effet du nombre de Rayleigh sur le temps d'apparition est discuté et clairement expliqué. La relation entre le nombre de Rayleigh et le nombre d'onde calculée ici s'accorde remarquablement bien avec la théorie de l'amplification.

DAS EINSETZEN DER INSTABILITÄT IN EINER WAAGERECHTEN FLUIDSCHICHT AUFGRUND EINES TEMPERATURSPRUNGES

Zusammenfassung—Es wird das Einsetzen der Instabilität in einer Fluidschicht, welche einer plötzlichen Temperaturänderung ausgesetzt ist, mit einer modifizierten Version der Hypothese des eingefrorenen Zustandes untersucht. Die Annahme, daß für große Prandtl-Zahlen die Temperaturstörungen auf die effektive thermische Eindringtiefe beschränkt sind, führt zu einer erheblichen Vereinfachung in der Formulierung des Stabilitäts-Problems. Der Einfluß der Rayleigh-Zahl auf die Auslösezeit wird behandelt und einleuchtend erklärt. Die Beziehung zwischen der Rayleigh-Zahl und der hier berechneten Wellen-Zahl stimmt mit der bestehenden Verstärkungstheorie bemerkenswert gut überein.

ВОЗНИКНОВЕНИЕ НЕУСТОЙЧИВОСТИ В ГОРИЗОНТАЛЬНОМ СЛОЕ ЖИДКОСТИ ПРИ СТУПЕНЧАТОМ ИЗМЕНЕНИИ ТЕМПЕРАТУРЫ

Аннотация—Модифицированным методом, использующим гипотезу "замороженного" времени, проанализировано возникновение неустойчивости в слое жидкости при мгновенном изменении температуры поверхности. Формулировку проблемы устойчивости можно существенно упростить, предположив, что при больших значениях числа Прандтля температурные возмущения ограничены областью эффективной тепловой глубины. Рассмотрено и обосновано влияние числа Рэлея на время возникновения неустойчивости. Полученное в работе соотношение между числом Рэлея и волновым числом достаточно хорошо согласуется с существующей линейной теорией.